

Aspects of Information Theory in Curved Space^{*†}

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Abstract. The often-asked question whether space-time is discrete or continuous may not be the right question to ask: Mathematically, it is possible that space-time possesses the differentiability properties of manifolds as well as the ultraviolet finiteness properties of lattices. Namely, physical fields in space-time could possess a finite density of degrees of freedom in the following sense: if a field's amplitudes are given on a sufficiently dense set of discrete points then the field's amplitudes at all other points of the manifold are fully determined and calculable. Which lattice of sampling points is chosen should not matter, as long as the lattices' spacings are tight enough, for example, not exceeding the Planck distance. This type of mathematical structure is known within information theory, as sampling theory, and it plays a central role in all of digital signal processing.

1 Introduction

Let us reconsider why the unification of general relativity and quantum theory has proven so difficult. Mathematically, the problems clearly begin with the fact that the two theories are formulated in the quite different languages of differential geometry and functional analysis. Physically, an important problem appears to be that general relativity and quantum theory, when considered together, are indicating that the notion of distance loses operational meaning at the Planck scale of about $10^{-35}m$ (assuming 3+1 dimensions). Namely, if one tries to resolve a spatial structure with an uncertainty of less than a Planck length, then the corresponding momentum uncertainty should randomly curve and thereby significantly disturb the very region in space that is meant to be resolved.

One of the problems in the effort of finding a unifying theory of quantum gravity is, therefore, to develop a mathematical framework which combines differential geometry and functional analysis such as to give a precise description of a notion of a shortest distance in nature. Candidate theories may become testable when introduced to inflationary cosmology and compared to the CMB measurements, see [1].

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In the literature, there has been much debate about whether the unifying theory will describe space-time as being discrete or continuous. It is tempting, also, to speculate that a quantum gravity theory such as M theory, see e.g. [2], a noncommutative geometric theory, see e.g. [3], or a foam theory, see e.g. [4], once fully understood, might reveal the structure of space-time as being in some sense in between discrete and continuous, possibly such as to combine the differentiability of manifolds with the ultraviolet finiteness of lattices. At first sight, this third possibility seems to be ruled out, however: as Gödel and Cohen proved, no set can be explicitly constructed whose cardinality would be in between discrete and continuous, see e.g. [5].

The message of this talk is that, nevertheless, there still is at least one mathematical possibility by which a theory of quantum gravity might yield a description of space-time which combines the differentiability of manifolds with the ultraviolet finiteness of lattices:

2 Fields with a finite density of degrees of freedom

Let us recall that physical theories are formulated not directly in terms of points in space or in space-time but rather in terms of the functions in space or in space-time. This suggests a whole new class of mathematical models for a finite minimum length.

Namely, fields in space-time could be functions over a differentiable manifold as usual, while, crucially, the class of physical fields is such that if a field is sampled only at discrete points then its amplitudes can already be reconstructed at *all* points in the manifold - if the sampling points are spaced densely enough. The maximum average sample spacing which allows one to reconstruct the continuous field from discrete samples could be on the order of the Planck scale, see [6].

Since any one of all sufficiently tightly spaced lattices would allow reconstruction, no particular lattice would be preferred. It is because no particular lattice is singled out that the symmetry properties of the manifold can be preserved.

The physical theory, i.e. fields and actions etc. could be written, equivalently, either as living on a differentiable manifold, thereby displaying e.g. external symmetries, or as living on any one of the sampling lattices of sufficiently small average spacing, thereby displaying its ultraviolet finiteness. Physical fields, while being continuous or even differentiable, would possess only a finite density of degrees of freedom.

The mathematics of classes of functions which can be reconstructed from discrete samples is well-known, namely as *sampling theory*, in the information theory community, where it plays a central role in the theory of sources and channels of continuous information as developed by Shannon, see [7].

3 Sampling theory

The simplest example in sampling theory is the Shannon sampling theorem: Choose a frequency ω_{max} . Consider the class $B_{\omega_{max}}$ of continuous functions f whose frequency content is limited to the interval $(-\omega_{max}, \omega_{max})$, i.e. for which: $\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} = 0$ whenever $|\omega| \geq \omega_{max}$. If the amplitudes $f(x_n)$ of such a function are known at equidistantly spaced discrete values $\{x_n\}$ whose spacing is π/ω_{max} or smaller, then the function's amplitudes $f(x)$ can be reconstructed for all x . The reconstruction formula is:

$$f(x) = \sum_{n=-\infty}^{\infty} f(x_n) \frac{\sin[(x - x_n)\omega_{max}]}{(x - x_n)\omega_{max}} \quad (1)$$

The theorem is in ubiquitous use in digital audio and video as well as in scientific data taking. Sampling theory, see [8], studies generalizations of the theorem for various different classes of functions, for non-equidistant sampling, for multi-variable functions and it investigates the effect of noise, which could be quantum fluctuations in our case. As was shown in [6], generalized sampling theorems automatically arise from stringy uncertainty relations, namely whenever there is a finite minimum position uncertainty Δx_{min} , as e.g. in uncertainty relations of the type: $\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta(\Delta p)^2 + \dots)$, see [9]. A few technical remarks: the underlying mathematics is that of symmetric non self-adjoint operators. Through a theorem of Naimark, unsharp variables of POVM type arise as special cases.

4 Information Theory on Curved Space

Let us consider as a natural (because covariant) analogue of the bandwidth restriction of the Shannon sampling theorem in curved space the presence of a cutoff on the spectrum of the Laplace operator $-\Delta$ on a Riemannian manifold \mathcal{M} (or the d'Alembert or the Dirac operator on a pseudo-Riemannian or a spin manifold respectively).

We start with the usual Hilbert space \mathcal{H} of square integrable scalar functions over the manifold, and we consider the dense domain $\mathcal{D} \subset \mathcal{H}$ on which the Laplacian is essentially self-adjoint. Using physicists' sloppy but convenient terminology we will speak of all points of the spectrum as eigenvalues, λ , with corresponding "eigenvectors" $|\lambda\rangle$. Since we are mostly interested in the case of noncompact manifolds, whose spectrum will not be discrete, some more care will be needed, of course. For Hilbert space vectors we use the notation $|\psi\rangle$, in analogy to Dirac's bra-ket notation, only with round brackets.

Let us define P as the projector onto the subspace spanned by the eigenspaces of the Laplacian with eigenvalues smaller than some fixed maximum value λ_{max} . (For the d'Alembertian and for the Dirac operator, let λ_{max} bound the absolute values of the eigenvalues.)

We consider now the possibility that in nature all physical fields are contained within the subspace $\mathcal{D}_s = P\mathcal{D}$, where λ_{max} might be on the order of $1/l_{Planck}^2$. In fact, through this spectral cutoff, each function in \mathcal{D}_s acquires the sampling property: if its amplitude is known on a sufficiently dense set of points of the manifold, then it can be reconstructed everywhere. Thus, through such a spectral cutoff a sampling theorem for physical fields arises naturally. To see this, assume for simplicity that one chart covers the N -dimensional manifold. Consider the coordinates \hat{x}_j , for $j = 1, \dots, N$ as operators that map scalar functions to scalar functions: $\hat{x}_j : \phi(x) \rightarrow x_j \phi(x)$. On their domain within the original Hilbert space \mathcal{H} , these operators are essentially self-adjoint, with an ‘‘Hilbert basis’’ of non-normalizable joint eigenvectors $\{|x\rangle\}$. We can write scalar functions as $\phi(x) = (x|\phi)$, i.e. scalar functions are the coefficients of the abstract Hilbert space vector $|\phi\rangle \in \mathcal{H}$ in the basis of the vectors $\{|x\rangle\}$. The continuum normalization of the $|x\rangle$ is with respect to the measure provided by the metric. On the domain of physical fields, \mathcal{D}_s , the multiplication operators \hat{x}_j are merely symmetric but not self-adjoint. The projections $P|x\rangle$ of the eigenvectors $|x\rangle$ onto the physical subspace \mathcal{D}_s are in general no longer orthogonal. Correspondingly, the uncertainty relations are modified, see [10].

Consider now a physical field, i.e. a vector $|\phi\rangle \in \mathcal{D}_s$, which reads as a function: $\phi(x) = (x|\phi)$. Assume that only at the discrete points $\{x_n\}$ the field’s amplitudes $\phi(x_n) = \langle x_n|\phi\rangle$ are known. Then, if the discrete sampling points $\{x_n\}$ are sufficiently dense, they fully determine the Hilbert space vector $|\phi\rangle$, and therefore $\phi(x)$ everywhere. To be precise, we assume the amplitudes

$$\phi(x_n) = (x_n|\phi) = \sum_{|\lambda| < \lambda_{max}}^f (x_n|\lambda)(\lambda|\phi) d\lambda \quad (2)$$

to be known. We use the sum and integral notation because $\{\lambda\}$ may be discrete and or continuous (the manifold \mathcal{M} may or may not be compact). Define $K_{n\lambda} = (x_n|\lambda)$. The set of sampling points $\{x_n\}$ is dense enough for reconstruction iff K is invertible, because then: $(\lambda|\phi) = \sum_n K_{\lambda,n}^{-1} \phi(x_n)$ and we therefore obtain the reconstruction formula:

$$\phi(x) = \sum_n \left(\sum_{|\lambda| < \lambda_{max}}^f (x|\lambda) K_{\lambda n}^{-1} d\lambda \right) \phi(x_n) \quad (3)$$

In communication theory, the stability of the reconstruction is important due to noise and is handled as in [11]. Here, not only may quantum fluctuations act as ‘noise’, but information can also be entangled. Still, following Shannon and Landau, it is natural to define the density of degrees of freedom through the number of dimensions of the space of functions in \mathcal{D}_s with essential support in a given volume. Clearly, we recover conventional Shannon sampling as a special case. The Shannon case has been applied to inflationary cosmology in [1] for flat space. It should be very interesting to apply to cosmology also the general approach presented here, both to generic non-flat spatial slices, and

also to the fully covariant case based on a cutoff of the spectrum of the Dirac or d'Alembert operator. In particular, the analysis of the analog of sampling theory in the case of indefinite metrics should provide a new approach to the problem of generally covariant UV cutoffs.

We also note that higher than second powers of the fields (second powers occur as scalar products in the Hilbert space of fields) are now nontrivial in quantum field theoretical actions: This is because the multiple product of fields needs to be defined such as to yield a result within the cut-off Hilbert space. In this context, it should be interesting also to reconsider the mechanism of Sakharov's induced gravity, see [12].

A sampling theoretical cutoff can be applied in arbitrary dimensions and it should be of interesting, e.g., to model a maximum achievable information density on black hole horizons this way. For holography, see e.g. [13]. Note that the sampling theoretical cutoff is always holographic in the sense that all information is encoded already in zero-dimensional sets, namely in any set of sampling points from which reconstruction is possible. In principle, holography in this sense should not be surprising: Any quantum theory which lives on a separable Hilbert space, in any space-time dimension, lives on a Hilbert space with a countable basis. For example, the Hilbert space of ordinary QM in three dimensions is unitarily equivalent to the Hilbert space of QM in any other number of dimensions, simply because all separable Hilbert spaces are unitarily isomorphic. The key observation in our sampling theory approach is that discrete sets of formal position eigenvectors can be chosen as such a countable basis in the Hilbert space, if there is bandwidth cutoff.

Our approach to sampling on curved space significantly simplifies in the case of compact manifolds, where the spectrum of the Laplacian is discrete and the cut off Hilbert space \mathcal{H}_s is finite dimensional. Intuitively, it is clear that knowledge of a function at as many points as is the dimension of the cutoff Hilbert space generically allows one to reconstruct the function everywhere. If the compact manifold is a group, the Peter-Weyl theorem provides us explicitly with the finite-dimensional Hilbert spaces of functions of "finite bandwidth" obtained by cutting of the spectra of Casimir operators. In the particular case of $SU(2)$ and the Laplacian we obtain the fuzzy sphere, see e.g. [14], which has been much discussed in the context of noncommutative geometry, [3].

In the literature, sampling theory on generic Riemannian manifolds has been little studied so far. This is because sampling theory originated and finds most of its applications in communication engineering. Interesting results that are of relevance here were obtained, however, by Pesenson, see e.g. [15], who considered, in particular, the case of homogeneous manifolds. In [15], the starting point is also a cutoff on the Laplace operator's spectrum. Reconstruction, however, works differently, namely by approaching the solution iteratively in a Sobolev space setting.

Much about the Laplace operator has been established in the field of spectral geometry, see e.g. [16, 17, 18, 19] and these methods and results should be very useful for our approach described here.

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